Analytically Determined Quasi-Static Parameters of Shielded or Open Multiconductor Microstrip Lines

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***Abstract*— An exact analytical expression for the capacitance matrix of a shielded or open multiconductor microstrip structure is derived by solving the system's dual integral equations by constructing a Volterra boundary-value problem (BVP). The solution is expressed in terms of infinite matrices with very good convergence properties. This new approach uses a series of Bessel functions rather than trigonometric series to approximate the solution which results in an efficient algorithm. Simplified formulas are given for the even and odd capacitance of symmetric coupled microstrip lines and compared to the results given by the finite analytical solution available in this particular case. Numerical examples demonstrate that the method yields accurate results and is computationally effective for structures having a large number of conductors.**

***Index Terms*— Boundary-value problems, integral equations, microstrip, multiconductor transmission lines.**

I. Introduction

MICROSTRIP transmission lines have received much attention in the technical literature in the last 30 years. Most of the efforts were dedicated to the analysis and electrical characterization of single or coupled microstrip lines. The continual increase of system clock speed in integrated circuit technology has made multiconductor transmission-lines analysis a vital tool not only for microwave applications, e.g., parallel coupled filters and directional couplers, but now also for high-speed digital integrated circuits (IC's).

Let us briefly review some theoretical methods relating to single and multiconductor microstrip lines. Wheeler [1] used approximate conformal mapping and an interpolation technique to calculate the capacity of inhomogeneous single microstrip lines, and Wan [17] used a similar method to accurately determine the quasi-static parameters of coupled microstrip lines. Analytical or quasi-analytical solutions have been provided for a limited number of cases and for certain particular geometries [8]–[10]. Bryant and Weiss [2] treated the problem by the method of moments and dielectric Green's function. Yamashita and Mittra [3] presented an analysis based on a variational principle. Analysis of various planar transmission lines have also been carried out in the spectral domain by Itoh and Mittra [4].

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Fikioris *et al.* [5] have given an exact solution for the shielded printed microstrip lines by the Carleman–Vekua method. Drake *et al.* [6] presented an improved spectral-domain method for coplanar multiconductor boxed microstrip lines. In [7], an analytical method was given for determining the capacitance matrix of boxed multiconductor planar and cylindrical lines. In [8], an analytical solution is given to the single shielded or open microstrip problem based on a special representation formula.

This paper presents a new analytical determination for the capacitance matrix of the planar shielded or open multiconductor microstrip lines embedded in a multilayered medium. This paper expands on the treatment in [7], where only a limited lateral structure was considered. As a consequence of the domain limitation, the electrical potential was expressed as a Fourier series. The boundaries conditions led to a dual series equation system, which eventually was transformed into a Volterra problem. This paper addresses the more difficult problem of the unlimited lateral domains. The electrical potential is expressed as a Fourier integral of the spectral-domain potential and the boundary conditions lead to a system of dual integral equations. The unknown spectral function is developed into a special sum of weighted Bessel functions. This sum fulfills exactly the boundary conditions everywhere except at a finite interval. Some subsequent transformations leads to a Volterra boundary-value problem (BVP). The particular expansion permits a drastic reduction of the number of terms necessary to be taken into account in order to obtain an accurate solution. Due to the reduced dimension of the system, this method is more efficient than the previous method for boxed structures. This method can be extended to support multilayer structures and noncoplanar strips as was done in [7]. Due to the similarity, the extension is not presented.

Section II describes the geometry of the shielded multiconductor microstrip line. The problem is then formulated as a dual integral equation system, whose solution is expressed by a special representation formula, which transforms the mentioned system into an infinite system of algebraic equations. This solution yields the Maxwell capacitance matrix of the multiconductor microstrip line. Simplified relations are then given for the even and odd capacitance of symmetric coupled microstrip. Our simplified relations are compared to an analytical solution of a particular case in Section IV. Also addressed is the computation time for our present method versus that of [7], as well as a comparison between the solutions obtained by our present method with those obtained in some previously published works.

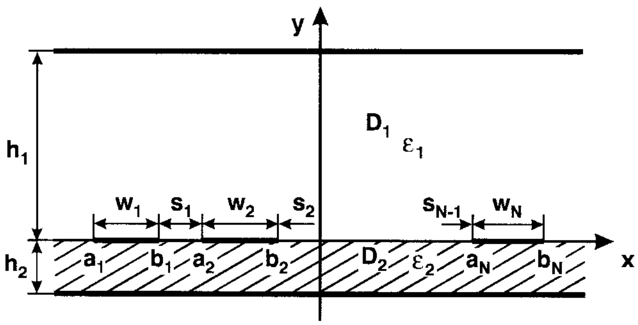

Fig. 1. Cross section of the multicondutor microstrip structure.

II. The Geometry of the Problem

The cross section of the shielded multiconductor microstrip line to be analyzed in this paper is shown in Fig. 1. It consists of N conducting strips of zero thickness $[a_j, b_j]$, $(j = 1, \cdots, N)$, with finite widths $w_1, \cdots, w_N$, separated by $N-1$ finite gaps $s_1, \cdots, s_{N-1}$, laying on a dielectric substrate between two parallel ground planes. The structure is considered infinite in the x-direction. The scale and the origin are taken to make $a_1 = -1$ and $b_N = 1$. The relative dielectric constants ϵ_1 and ϵ_2 and the dielectric thickness h_1 and h_2 of domains D_1 and D_2 are arbitrary. In particular, for $h_1 \to \infty$ the open multiconductor microstrip is obtained.

We give the solution for the multiconductor microstrip problem considering only two dielectric substrates.

III. Analysis of the Multiconductor Microstrip Line

We consider the solution of the microstrip problem in the "quasi-static" approximation, i.e., for the frequency range in which propagation may be regarded as quasi-TEM. Under this approximation, the analysis of a multiconductor microstrip line is reduced to the determination of the Maxwell capacitance matrix per unit length. The full-wave solution can be obtained based on the "quasi-static" solution proposed in this paper. The authors will present a full wave solution in a future paper.

In the quasi-TEM assumption, the electric field in domains D_1 and D_2 can be expressed with the aid of the electrostatic potential $V^{(1)}(x, y)$ and $V^{(2)}(x, y)$ as follows:

$$V^{(1)}(x, y) = \int_{-\infty}^{\infty} A(k) \frac{\sinh k(h_1 - y)}{\sinh kh_1} \exp(ikx)\, dk \quad (1)$$

$$V^{(2)}(x, y) = \int_{-\infty}^{\infty} A(k) \frac{\sinh k(h_2 + y)}{\sinh kh_2} \exp(ikx)\, dk. \quad (2)$$

The functions $V^{(1)}(x, y)$ and $V^{(2)}(x, y)$ vanish on the planes $y = h_1$ and $y = -h_2$, respectively, and satisfy the potential continuity condition on the plane $y = 0$. For $y = 0$ and $x \in (a_j, b_j)$, the potential is constant due to the conducting strip, and the first integral equation of the problem is

$$\int_{-\infty}^{\infty} A(k) \exp(ikx)\, dk = V_j, \qquad \begin{array}{l} x \in (a_j, b_j) \\ j = 1, \cdots, N. \end{array} \quad (3)$$

On each unit length of the jth strip there is a charge q'_j and the sum of all charges is denoted $q_0 = \sum_{j=1}^{N} q'_j$. An integration with respect to x of the charge density in the plane $y = 0$ yields an expression representing the total charge, up to an additive constant. The condition that the total charge must be constant for x corresponding to the gaps give the second integral equation

$$\int_{-\infty}^{\infty} A(k)[\epsilon_1 \coth(kh_1) + \epsilon_2 \coth(kh_2)] \frac{\exp(ikx)}{i}\, dk$$
$$= q(x) = \begin{cases} q_0/2 \operatorname{sgn} x, & x \notin (a_1, b_N) \\ q_j = -q_0/2 + \sum_{l=1}^{j} q'_l, & x \in (b_j, a_j + 1) \\ & j = 1, \cdots, N-1 \end{cases} \quad (4)$$

where $\operatorname{sgn} x = -1$ for $x < 0$ and $\operatorname{sgn} x = 1$ for $x > 0$.

We put

$$A(k) \equiv \frac{B(k)}{\epsilon_1 + \epsilon_2}[1 - \eta(k)] \quad (5)$$

where

$$\eta(k) = \frac{\epsilon_1 (\coth(kh_1) - 1) + \epsilon_2 (\coth(kh_2) - 1)}{\epsilon_1 \coth(kh_1) + \epsilon_2 \coth(kh_2)}. \quad (6)$$

The function $\eta(k)$ has an exponentially decay to zero for large values of the variable k. As it has been shown in [7], the multilayer structures can be analyzed by this method modifying only the function $\eta(k)$.

We separate the real and the imaginary part of $B(k)$ $(B(k) = B'(k) - iB''(k))$ and we take into account the evenness of the real part and the oddness of the imaginary part of $B(k)$.

Hence, (3) and (4) become

$$\int_0^{\infty} [B'(k) \cos kx + B''(k) \sin kx]\, dk = (\epsilon_1 + \epsilon_2) V_j$$
$$+ \int_0^{\infty} [B'(k) \cos kx + B''(k) \sin kx] \eta(k)\, dk \quad (7)$$

$$\int_0^{\infty} [B'(k) \sin kx - B''(k) \cos kx]\, dk = q(x). \quad (8)$$

Relations (7) and (8) are the system of dual integral equations which solve the problem.

We associate to any point a_j the angle $\alpha_j = \arcsin a_j$, and the complex point $c_j = \exp(i\alpha_j)$, and to any point b_j the angle $\beta_j = \arcsin b_j$, and the complex point $d_j = \exp(i\beta_j)$.

We look for the solution of the dual integral equations system (7) and (8) in the following form:

$$B'(k) = \frac{q_0}{\pi} \left[\frac{J_0(k)}{k} + 2c\delta(k) \right] + \sum_{m=1}^{\infty} 2m b'_{2m} \frac{J_{2m}(k)}{k} \quad (9)$$

$$B''(k) = + \sum_{m=1}^{\infty} (2m-1) b'_{2m-1} \frac{J_{2m-1}(k)}{k} \quad (10)$$

where $\delta(k)$ is the Dirac function, $c = 0.577\,216$ is the Euler constant, and b'_{2m} and b''_{2m-1} are real coefficients to be determined.

The substitution of (9) and (10) in (7) followed by the use of (31)–(34) gives

$$\sum_{m=1}^{\infty} b'_{2m} \cos 2m\phi + \sum_{m=1}^{\infty} b''_{2m-1} \sin(2m-1)\phi = f_j(\phi) \quad (11)$$

where $\phi = \arcsin x$, $\phi \in (\alpha_j, \beta_j)$, $j = 1, \cdots, N$ and

$$f_j(\phi) = (\epsilon_1 + \epsilon_2)V_j - \frac{q_0}{\pi}\ln 2 + \int_0^\infty [B'(k)\cos(k\sin\phi) + B''(k)\sin(k\sin\phi)]\eta(k)\,dk. \quad (12)$$

Proceeding in the same way with (8), one obtains the following for $|x| < 1$, $(\phi \in (-\pi/2, \pi/2))$:

$$\sum_{m=1}^{\infty} b'_{2m}\cos 2m\phi + \sum_{m=1}^{\infty} b''_{2m-1}\sin(2m-1)\phi = g_j(\phi) \quad (13)$$

where $\phi \in (\beta_j, \alpha_{j+1})$, $j = 1, \cdots, N-1$, $g_j(\phi) = q_j - \frac{q_0}{2}\phi$. For $|x| > 1$, (8) is identically satisfied.

The right-hand term of (12) is expanded using the following relations:

$$\cos(k\sin\phi) = J_0(k) + 2\sum_{m=1}^{\infty} J_{2m}(k)\cos 2m\phi \quad (14)$$

$$\sin(k\sin\phi) = 2\sum_{m=1}^{\infty} J_{2m-1}(k)\sin(2m-1)\phi. \quad (15)$$

The final result of this expansion can be written as

$$f_j(\phi) = (\epsilon_1 + \epsilon_2)V_j - \frac{q_0}{\pi}\ln 2 + \frac{b'_0}{2} + \sum_{m=1}^{\infty}\left(\frac{q_0}{\pi}\frac{a_{0,\,2m}}{2m} + \sum_{n=1}^{\infty} b'_{2n}a_{2m,\,2n}\right)\cos 2m\phi + \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} b'_{2n-1}a_{2m-1,\,2n-1}\right)\sin(2m-1)\phi. \quad (16)$$

In (16) we have denoted

$$b'_0 = q_0 a_{0,\,0} + \sum_{n=1}^{\infty} a_{0,\,2n} b'_{2n} \quad (17)$$

$$a_{0,\,0} = \frac{2}{\pi}\int_0^\infty \frac{J_0^2(k) - \exp(-k)}{k}\eta(k)\,dk \quad (18)$$

$$a_{m,\,n} = 2n\int_0^\infty \frac{J_n(k)J_m(k)}{k}\eta(k)\,dk, \qquad m = 0, 1, 2, \cdots \quad n = 1, 2, \cdots. \quad (19)$$

The effective evaluation of integrals (18)–(19) is done numerically by using the Gauss–Laguerre formula [9].

For the time being, let the functions $f_j(\phi)$ and $g_j(\phi)$ be considered as known. We introduce the complex variable function

$$F(z) = \sum_{m=1}^{\infty} b'_{2m} z^{2m} + \sum_{m=1}^{\infty} (-i) b''_{2m-1} z^{2m-1} \quad (20)$$

defined in the domain $|z| < 1$. One can verify that for $z = \exp(i\phi)$ (11) and (13) can be written as

$$\begin{cases} \mathrm{Re}\{F(\exp(i\phi))\} = f_j(\phi), & \phi \in (\alpha_j, \beta_j) \\ \mathrm{Im}\{F(\exp(i\phi))\} = g_j(\phi), & \phi \in (\beta_j, \alpha_{j+1}). \end{cases} \quad (21)$$

By construction, $F(z)$ takes only real values on the imaginary axis and according to the symmetry principle [15] we have

$$F(\overline{z}) = \overline{F(z)} \quad (22)$$

where the overline denotes the complex conjugate quantities.

The relations of (21) yield the boundary values for the analytic function $F(z)$ along half of the unit circle ($\phi \in [-\pi/2, \pi/2]$), either the real part (for the small arches between c_j and d_j, $j = 1, \cdots, N$), or the imaginary part (for the small arches between d_j and c_{j+1}, $j = 1, \cdots, N-1$). The boundary values for the other half of the unit circle are obtained using (22) and the Volterra problem is completely defined.

In order to transform the Volterra problem into a Dirichlet problem, we consider the auxiliary complex function

$$H(z) = \prod_{j=1}^{N-1}\sqrt{\frac{z - c_{j+1}}{z - d_j}\frac{z + \overline{c}_{j+1}}{z + \overline{d}_j}}. \quad (23)$$

This function has along the unit circle either real values of imaginary values. By its construction $H(z)$ takes real values on the arches where we know the real part of $F(z)$ and imaginary values where we know the imaginary part of $F(z)$.

We now define a new unknown function $G(z) = F(z)/H(z)$. The real part of this function is known along the entire unit circle. This is a Dirichlet BVP for the real part of the complex function $G(z)$. The complex function $G(z)$ can be obtained by means of Schwartz's formula [11], [16], shown in (24), at the bottom of the page, where $z' = \exp(\arg z')$, $\mathcal{A}_j$ is the symbol for the union of the arches (c_j, d_j) and $(-\overline{d}_j, -\overline{c}_j)$, whereas $\mathcal{B}_j$ is the symbol for the union of arches (d_j, c_{j+1}) and $(-\overline{c}_{j+1}, -\overline{d}_j)$.

The unknown coefficients b'_{2m} and b''_{2m-1} are the MacLaurin expansion coefficients of the function $F(z)$. The infinite linear system for their determination results by matching the coefficient of z^n; $(n = 1, 2, \cdots)$ on the two sides of (24). Adding (17), we write the system in the matrix form

$$\mathbf{X}\langle b\rangle = \mathbf{Y}\langle v\rangle + \mathbf{Z}\langle q\rangle \quad (25)$$

where $\langle b\rangle^t = (b'_0, b''_1, b'_2, b''_3, \cdots)$, $\langle v\rangle^t = (V_1, \cdots, V_N)$, $\langle q\rangle^t = (q'_1, \cdots, q'_N)$. The linear system of (25) is developed in Appendix B.

In order to compensate for the singularities of the function $H(z)$ in the d_j points, the expression in braces in (24) must vanish in all these points. Hence, we obtain the following

$$F(z) = \left\{\frac{H(z)}{i\pi}\sum_{j=1}^{N}\int_{\mathcal{A}_j}\frac{f_j(\arg z')}{|H(z')|(z'-z)}dz' + \sum_{j=1}^{N-1}\int_{\mathcal{B}_j}\frac{g_j(\arg z')}{|H(z')|(z'-z)}\,dz'\right\} \quad (24)$$

TABLE I

s/h	0.1		0.2		0.5		1		2	
w/h	C_{even}	C_{odd}	C_{even}	C_{odd}	C_{even}	C_{odd}	C_{even}	C_{odd}	C_{even}	C_{odd}
0.1	1.0849	3.1379	1.1644	2.5766	1.3214	2.0258	1.4643	1.7599	1.5688	1.6291
	1.0861	3.1377	1.1645	2.5766	1.3214	2.0258	1.4643	1.7599	1.5688	1.6290
0.2	1.3343	3.8332	1.4167	3.1708	1.5962	2.4857	1.7708	2.1459	1.9021	1.9787
	1.3342	3.8333	1.4167	3.1708	1.5962	2.4857	1.7708	2.1460	1.9023	1.9791
0.5	1.9717	4.9694	2.0572	4.2177	2.2625	3.3886	2.4777	2.9573	2.6445	2.7427
	1.9717	4.9693	2.0572	4.2175	2.2625	3.3885	2.4777	2.9576	2.6449	2.7451
1	2.9781	6.1723	3.0657	5.3922	3.2825	4.5105	3.5158	4.0414	3.6984	3.8061
	2.9781	6.1697	3.0670	5.3891	3.2825	4.5065	3.5153	4.0379	3.6963	3.8095
2	4.9786	8.2185	5.0669	7.4323	5.2866	6.5388	5.5242	6.0609	5.7107	5.8207
	4.9786	8.1835	5.0668	7.3953	5.2861	6.4974	5.5206	6.0195	5.6904	5.7998

existence conditions:

$$\sum_{j=1}^{N}\int_{A_j}\frac{f_j\,(\arg z')}{|H(z')|(z'-d_k)}\,dz' + \sum_{j=1}^{N-1}\int_{\mathcal{B}_j}\frac{g_j\,(\arg z')}{|H(z')|(z'-d_k)}\,dz' = 0 \qquad (26)$$

where $k = 1,\cdots,N-1$. There are only $N-1$ distinct compatibility conditions. In fact, due to symmetry, if the expression in braces in (24) vanishes for $z = d_k$, it also vanishes for $z = -\overline{d}_k$.

All the $N-1$ compatibility conditions have the same real part, which is, in fact, an identity. The coefficient of z^0 in the MacLaurin expansion must be equal to $F(0) = 0$; this yields an additional compatibility condition besides (26). Hence, the compatibility conditions can be written in matrix form as

$$\mathbf{D}\langle b\rangle = \mathbf{A}\langle v\rangle + \mathbf{B}\langle q\rangle. \qquad (27)$$

Again, the linear system (27) is explicitly given in Appendix B.

If we eliminate the unknown infinite vector $\langle b\rangle$ between (25) and (27), we finally obtain the capacitance matrix of the given system as follows:

$$\mathbf{C} = (\mathbf{D}\mathbf{X}^{-1}\mathbf{Z} - \mathbf{B})^{-1}(\mathbf{A} - \mathbf{D}\mathbf{X}^{-1}\mathbf{Y}). \qquad (28)$$

This relation gives an exact expression of the capacitance matrix. To obtain a numerical estimation we must truncate the infinite matrices. For the important case of a symmetric microstrip coupled line ($w_1 = w_2 = w$), we derive the linear systems (29) for even and (30) for odd capacitance considering four terms in the expansion (20). The notations are those used in Appendix B:

$$C_{\text{even}} = -(\epsilon_1+\epsilon_2)[L_0 Ae_{1,1} + L_2 Ae_{1,2} + L_4 Ae_{1,3}]$$

$$Ae = -\left\{\begin{bmatrix} N_0 & 0 & 0 \\ N_2 & K(0) & 0 \\ N_4 & K(-2) & K(0)\end{bmatrix} - \begin{bmatrix} L_0 & O_{0,2} & O_{0,4} \\ L_2 & O_{2,2} & O_{2,4} \\ L_4 & O_{4,2} & O_{4,4}\end{bmatrix} \cdot \begin{bmatrix} a_{0,0} - 2\ln 2/2\pi & a_{0,2} & a_{0,4} \\ a_{0,2}/2\pi & a_{2,2} & a_{2,4} \\ a_{0,4}/4\pi & a_{4,2} & a_{4,4}\end{bmatrix}\right\}^{-1} \qquad (29)$$

$$C_{\text{odd}} = (\epsilon_1+\epsilon_2)[(Q_{1,1}-Q_{1,2})Ac_{1,1} + (L_{1,1}-L_{1,2})Ac_{1,2} + (L_{3,1}-L_{3,2})Ac_{1,3}]$$

$$Ac = \left\{\begin{bmatrix} 0 & U_{1,1} & U_{1,3} \\ 0 & P_{1,1} & P_{1,3} \\ 0 & P_{3,1} & P_{3,3}\end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{1,1} & a_{1,3} \\ 0 & a_{3,1} & a_{3,3}\end{bmatrix} + \begin{bmatrix} -R_{1,1} & 0 & 0 \\ -M_{1,1} & K(0) & 0 \\ -M_{3,1} & K(-2) & K(0)\end{bmatrix}\right\}. \qquad (30)$$

IV. Numerical Results

In order to evaluate the usefulness of (29) and (30), we consider the particular case of equal dielectric thicknesses ($h_1 = h_2 = h$) for which closed analytical expressions for the even and the odd capacitance are available [10]. The expression of the capacitance can be written as

$$C = (\epsilon_1+\epsilon_2)\frac{2K(k)}{K(\sqrt{1-k^2})}$$

$$k = \tanh\left(\frac{\pi w}{4h}\right) \otimes \tanh\left(\frac{\pi(w+s)}{4h}\right).$$

$K(k)$ is the complete elliptical integral of the first kind. $\otimes$ is either the multiplication operator $(\cdot)$ for the even case or the division operation (/) for the odd case.

The results for various values of w/h and s/h are reported in Table I ($\epsilon_1 = 1$ and $\epsilon_2 = 1$). In any cell the first row

TABLE II

	C_{11}	C_{22}	C_{33}	C_{44}	C_{12}	C_{23}	C_{34}	C_{45}	C_{13}	C_{24}
∞	14.448	17.556	17.705	17.730	-6.6119	-5.9398	-5.8759	-5.8653	-1.4740	-1.1829
e	14.451	17.556	17.705	17.731	-6.6104	-5.9393	-5.8755	-5.8650	-1.4730	-1.1823
m	14.445	17.555	17.704	17.730	-6.6134	-5.9404	-5.8762	-5.8656	-1.4752	-1.1832
	C_{35}	C_{14}	C_{25}	C_{36}	C_{15}	C_{26}	C_{16}	C_{27}	C_{17}	C_{18}
∞	-1.1503	-0.6477	-0.4922	-0.4769	-0.3522	-0.2619	-0.2147	-0.1634	-0.1456	-0.1383
e	-1.1500	-0.6469	-0.4918	-0.4765	-0.3515	-0.2615	-0.2140	-0.1629	-0.1448	-0.1369
m	-1.1507	-0.6486	-0.4927	-0.4773	-0.3531	-0.2623	-0.2155	-0.1639	-0.1466	-0.1400

corresponds to the closed formulas, the second to (29) and (30) solutions. As it can be seen from Table I, for $0.1 < s/h$, $w/h < 2$ the error in the approximate C_{even} is less than 0.5% while the error in C_{odd} is less than 0.7%.

The multiconductor structure with the parameters $N = 8$, $\epsilon_1 = 12.9$, $\epsilon_2 = 1$, $h_1 = 16$, $h_2 = 100$, $w_1 = \cdots = w_8 = 1$, $s_1 = \cdots s_7 = 1$ has been analyzed by the present method and by that proposed in [7] for multiconductor boxed structures. In order to apply the method developed in [7] we considered electric and magnetic lateral walls, the minimum space between the conductors and the lateral wall being five times the lower dielectric thickness ($s_0 = s_8 = 80$). The results for the 22 different coefficients of the capacitance matrix are reported in Table II. The electric lateral-wall case coefficients are always higher than the unlimited lateral case coefficients, and both cases give higher coefficients than the magnetic lateral-wall case which is an expected result.

In order to compare the computer efficiency of the present method and that of [7] we compare the time necessary to attain a certain accuracy. Let us denote NT: the number of terms in the development of the auxiliary function $F(z)$ (20), ND: the number of the sampling points for the line integrals (36)–(38) and (43)–(45) NF: the number of points for the fast Fourier transform (FFT) used to compute the integrals (39)–(41), (46), and (47). The computation time and the precision diminish if these numbers decrease. First, by using higher values for NT, ND, and NF, an asymtotic capacitance matrix is obtained. Next, NT is reduced until the highest relative error in any terms is just smaller than 10^{-5}, i.e., if NT is decremented once more the relative error for at least one of the capacitance matrix coefficients would be higher than 10^{-5}. We proceeded in the same way with ND and NF. The results are given in Table III.

Under these conditions, using a PC with a Cyrix P-120 processor, the mean computation time was 0.423 s for the MATLAB implementation of the present algorithm, versus 2.36 s (also in MATLAB) for the other approach. From a practical point of view the results are equivalent, but the computation time is reduced more than five times. This result suggests that the present approach can be used advantageously for boxed structure with distant lateral walls.

We have considered the case of a multistrip structure ($N = 5$, $\epsilon_1 = 1$, $\epsilon_2 = 1$, $h_1 = 5$, $h_2 = 5$, $w_1 = \cdots = w_5 = 2$, $s_1 = \cdots = s_4 = 1$) studied by Kammler in [13]. The results obtained using ten terms in the expansion (20) are shown below and are identical within the first five digits with the numerical results given in [13]

TABLE III

	Present Method		Method from [2]	
Parameter	Initial	Final	Initial	Final
NT	20	8	40	22
ND	64	8	64	44
NF	512	64	512	128

$$C_{11} = C_{55} = 2.891\,43 \quad C_{13} = C_{35} = -0.079\,42$$
$$C_{22} = C_{44} = 3.293\,87 \quad C_{14} = C_{25} = -0.011\,74$$
$$C_{33} = 3.296\,09 \quad C_{24} = -0.075\,12$$
$$C_{12} = C_{45} = -1.006\,08 \quad C_{15} = -0.001\,97$$
$$C_{23} = C_{34} = -0.976\,38 \quad C_{ij} = C_{ji}.$$

In Table IV, the even- and odd-mode characteristic impedances obtained by the present method for a symmetric coupled microstrip line ($N = 2$, $\epsilon_1 = 2.35$, $\epsilon_2 = 1$, $h_1 = 1$, $h_2 = \infty$, $s = 1$, $w_1 = w_2 = w$) are compared with those obtained in [14] and [17]. The match is very good for the odd-mode characteristic impedance and good for the even-mode impedance. The results for the even-mode impedance are placed between those obtained in [14] and [17]. The even-mode impedance in [17] is always higher than the results obtained by the present method. The fact can be related to the assumption made in [17] that the slot between strips is a magnetic wall lowering the capacitance and increasing the characteristic impedance.

V. Conclusions

A new method for determining the Maxwell capacitance of a multiconductor microstrip structure is given. The method is based on solving a system of dual integral equations using a special representation formula by means of a Volterra BVP. An exact solution is expressed in terms of infinite matrices

TABLE IV

	Present Method		Wan		Hammerstad and Jensen	
w/h	Z_{even}	Z_{odd}	Z_{even}	Z_{odd}	Z_{even}	Z_{odd}
0.05	260.83	198.985	262.30	198.62	255.07	200.35
0.10	227.093	168.415	228.83	168.17	224.14	168.43
0.25	180.567	129.94	182.43	129.86	179.65	129.73
0.50	143.599	102.91	145.33	102.94	143.43	102.87
0.75	121.631	88.027	123.18	88.10	121.69	88.01
1.00	106.247	77.891	107.62	77.98	106.37	77.85
1.25	94.638	70.29	95.85	70.39	94.75	70.22
1.75	78.026	59.341	78.97	59.43	78.09	59.25
2.25	66.582	51.661	67.34	51.84	66.59	51.56

which have very good convergence properties. The numerical examples considered in this paper demonstrate that in practical cases it is sufficient to consider only a low number of terms in the representation formula for accurate results. Moreover, our solution is shown to be computationally efficient for the case of multiconductor lines.

Appendix A

In [8], we have demonstrated the subsequent formulas:

$$\int_0^\infty \left[\frac{J_0(k)}{k}+2c\delta(k)\right]\cos kx\,dk = \ln\frac{2}{b}, \qquad |x|<1 \tag{31}$$

$$\int_0^\infty \left[\frac{J_0(k)}{k}+2c\delta(k)\right]\sin kx\,dk = \begin{cases}\frac{\pi}{2}\,\mathrm{sgn}\,x, & |x|>1\\ \arcsin x, & |x|<1.\end{cases} \tag{32}$$

From [12], we have the following:

$$\int_0^\infty \frac{J_{2p-1}(k)}{k}\cos kx\,dk = \begin{cases}\frac{1}{2p-1}\cos\left((2p-1)\arcsin x\right), & |x|\le 1\\ 0, & |x|>1\end{cases} \tag{33}$$

$$\int_0^\infty \frac{J_{2p}(k)}{k}\sin kx\,dk = \begin{cases}1/2p\sin\left(2p\arcsin x\right), & |x|\le 1\\ 0, & |x|>1\end{cases} \tag{34}$$

where $p = 1, 2, \cdots$.

Appendix B

Here we give the computation relations involved in Section III. The MacLaurin expansion of the function $F(z)/H(z)$ must match the corresponding coefficients resulting in the expansion of the brace in (24) where we get the following:

$$\begin{aligned}&\sum_{n=1}^{E(p/2)} K(p-2n)b'_{2n} - i\sum_{n=1}^{E((p+1)/2)} K(p+1-2n)b''_{2n-1}\\ &= (\epsilon_1+\epsilon_2)\sum_{j=1}^{N} L_{p,\,j}V_j + \sum_{j=1}^{N-1}\left(M_{p,\,j}\sum_{l=1}^{j} q'_l\right)\\ &\quad + \left(-\frac{\ln 2}{\pi}L_p + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{a_{0,\,2m}}{2m}O_{p,\,2m}\right.\\ &\quad\left. - N_p - \sum_{j=1}^{N-1} M_{p,\,j}\right)\times\sum_{l=1}^{N} q'_l + b'_0 L_p\\ &\quad + \sum_{m=1}^{\infty}\left[\left(\sum_{n=1}^{\infty} b'_{2n}a_{2m,\,2n}\right)O_{p,\,2m}\right.\\ &\quad\left. + \left(\sum_{n=1}^{\infty} b'_{2n-1}a_{2m-1,\,2n-1}\right)P_{p,\,2m-1}\right].\end{aligned} \tag{35}$$

In (35), $E(x)$ is the integer part function and we have denoted the following:

$$L_{p,\,j} = \frac{1}{\pi}\int_{\mathcal{A}_j}\frac{dz'}{|H(z')|(z')^{p+1}}$$

$$L_p = \frac{1}{2}\sum_{j=1}^{N} L_{p,\,j} \tag{36}$$

$$M_{p,\,j} = \frac{1}{\pi}\int_{\mathcal{B}_j}\frac{dz'}{|H(z')|(z')^{p+1}} \tag{37}$$

$$N_p = \sum_{j=1}^{N-1}\frac{1}{\pi}\int_{\mathcal{B}_j}\frac{\frac{\arg z'}{\pi}\,dz'}{|H(z')|(z')^{p+1}} \tag{38}$$

$$K(r) = \frac{1}{2\pi}\oint_{\mathcal{C}_1}\frac{z^{r-1}}{H(z)}\,dz \tag{39}$$

$$O_{p,\,2m} = \frac{1}{2\pi}\sum_{j=1}^{N}\int_{\mathcal{A}_j}\frac{(z')^{2m}+(z')^{-2m}}{|H(z')|(z')^{p+1}}\,dz' \tag{40}$$

$$P_{p,\,2m-1} = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\mathcal{A}_j}\frac{(z')^{2m-1}-(z')^{-2m+1}}{|H(z')|(z')^{p+1}}\,dz'. \tag{41}$$

In (39), $\mathcal{C}_1$ is a circle inside the unit circle which contains the origin.

The compatibility conditions (26) can be written in the following form:

$$\begin{aligned}&(\epsilon_1+\epsilon_2)\sum_{j=1}^{N} Q_{k,\,j}V_j\sum_{j=1}^{N-1}\left(R_{k,\,j}\sum_{l=1}^{j} q'_l\right)\\ &\quad + \left(\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{a_{0,\,2m}}{2m}T_{k,\,2m} - S_k - \sum_{j=1}^{N-1} R_{k,\,j}\right)\sum_{l=1}^{N} q'_l\\ &\quad + \sum_{m=1}^{\infty}\left[\left(\sum_{n=1}^{\infty} b'_{2n}a_{2m,\,2n}\right)T_{k,\,2m}\right.\\ &\quad\left. + \left(\sum_{n=1}^{\infty} b'_{2n-1}a_{2m-1,\,2n-1}\right)U_{k,\,2m-1}\right] = 0.\end{aligned} \tag{42}$$

In (42), we have denoted the following:

$$Q_{k,\,j} = \mathrm{Im}\left(\frac{1}{\pi i}\int_{\mathcal{A}_j} \frac{dz'}{|H(z')|(z' - d_k)}\right) \tag{43}$$

$$R_{k,\,j} = \mathrm{Im}\left(\frac{1}{\pi i}\int_{\mathcal{B}_j} \frac{dz'}{|H(z')|(z' - d_k)}\right) \tag{44}$$

$$S_k = \sum_{j=1}^{N-1} \mathrm{Im}\left(\frac{1}{\pi i}\int_{\mathcal{B}_j} \frac{\frac{\arg z'}{\pi}\, dz'}{|H(z')|(z' - d_k)}\right) \tag{45}$$

$$T_{k,\,2m} = \mathrm{Im}\left(\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\mathcal{A}_j} \frac{(z')^{2m} + (z')^{-2m}}{|H(z')|(z' - d_k)}\, dz'\right) \tag{46}$$

$$U_{k,\,2m-1} = \mathrm{Im}\left(-\frac{1}{2\pi}\sum_{j=1}^{N}\int_{\mathcal{A}_j} \frac{(z')^{2m-1} - (z')^{-2m+1}}{|H(z')|(z' - d_k)}\, dz'\right). \tag{47}$$

Details about the numerical evaluation of the integrals (36)–(41) and (43)–(47) have been given in [7].

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